Note→ In any Boolean Algebra B, we have

(1) $(a')' = a \quad \forall a \in B$

(2) $0' = 1$ & $1' = 0$

Th→ <u>De Morgan's Laws</u>

For every pair of elements a & b in a Boolean Algebra B, we have—

$(a+b)' = a' * b'$; $(a*b)' = a' + b'$

Pf→ First we shall prove that—

$(a+b)' = a' * b'$

i.e. complement of $(a+b)$ is $a' * b'$

For this, we need to show that—

$(a+b) + (a' * b') = 1$ & $(a+b) * (a' * b') = 0$

; from defn of complement

$a + a' = 1$; $a * a' = 0$

Now→

$$
\begin{aligned}
(a+b)+(a' * b') &= (b+a) + (a' * b') && ;\ a+b = b+a \\
&= b + [a + (a' * b')] && ;\ \text{associativity} \\
&= b + [(a+a') * (a+b')] && ;\ \text{distr} \\
&= b + [1 * (a+b')] && ;\ a+a' = 1 \\
&= b + (a+b') && ;\ 1 * a = a \\
&= b + (b'+a) && ;\ \text{commutative} \\
&= (b+b') + a && ;\ \text{associativity} \\
&= 1 + a = 1 && ;\ \text{as } 1 + a = 1
\end{aligned}
$$

$\therefore (a+b) + (a' * b') = 1 \quad \longrightarrow (1)$

Now →

$$(a+b) * (a' * b') = [(a+b) * a'] * b' \text{ ; associativity}$$
$$= [(a*a') + (b*a')] * b' \text{ ; distr}$$
$$= [0 + (b*a')] * b' \text{ ; } a*a' = 0$$
$$= (b*a') * b' \text{ ; } 0+a = a$$
$$= (a'*b) * b' \text{ ; comm.}$$
$$= a' * (b*b') \text{ ; associativity}$$
$$= a' * 0 \text{ ; } b*b' = 0$$
$$= 0 \text{ ; } a*0 = 0$$

$\therefore (a+b) * (a' * b') = 0 \quad \longrightarrow (2)$

From (1) & (2), we get →

$$\boxed{(a+b)' = a' * b'} \longrightarrow (3)$$

Since duality principle holds in a Boolean Algebra, $\therefore$ from (3), we get →

$$\boxed{(a*b)' = a' + b'} \longrightarrow (4)$$

(3) & (4) are De Morgan laws.

(Proved)

Th 4.1 The following are equivalent in a Boolean Algebra B.

(i) $a+b=b$ (ii) $a*b=a$ (iii) $a'+b=1$

(iv) $a*b'=0$

Pf: (i) ⇒ (ii)

We have: $a = a+(a*b)$, absorption law

$= (a+a)*(a+b)$, distr

$= a*(a+b)$, idempotent

$= a*b$, $\because a+b=b$ given

(ii) ⇒ (i) Given $a*b=a$

To prove $a+b=b$

$a+b = (a*b)+b$, $\because a*b=a$ given

$= b+(a*b)$, comm

$= b+(b*a)$, comm

$= b$, absorption

(i) ⇒ (iii) Given $a+b=b$

To show $a'+b=1$

$a'+b = a'+(a+b)$, $\because a+b=b$ given

$= (a'+a)+b$, associative

$= 1+b$, as $a'+a=1$

$= 1$, boundedness

(ii) ⇒ (i) Given $a'+b=1$

To show $a+b=b$

we have-

$$a+b = 1 * (a+b)$$

$$= (a'+b) * (a+b), \quad \text{as } a'+b=1$$

$$= (a' * a) + b, \quad \text{dist}^{r}$$

$$= 0 + b, \quad a' * a = 0$$

$$= b$$

(iii) ⇒ (iv) Given $a'+b=1$

To show $a * b' = 0$

we have-

$$0 = 1'$$

$$= (a'+b)', \quad \text{as } 1 = a'+b \text{ given}$$

$$= (a')' * b', \quad \text{De-Morgan law}$$

$$0 = a * b'$$

(iv) ⇒ (iii) Given $a * b' = 0$

To show $a'+b=1$

we have-

$$1 = 0' = (a * b')', \quad 0 = a * b' \text{ given}$$

$$= a' + (b')', \quad \text{De-Morgan law}$$

$$= a'+b$$

$$1 = a'+b$$

(Proved)